Example Consider the matrices:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Describe each transformation in words.

Hint: Consider the letter L formed by the vectors $\mathbf{v} = (1, 0)$ and $\mathbf{v} = (0, 2)$, and let's call it our standard letter L. Examine how each of the matrices affects our standard letter L.

Scalings For any positive constant k, the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ defines a scaling by k, since

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\vec{x}.$$

This is a *dilation* (or enlargement) if k exceeds 1, and it is a *contraction* (or shrinking) for values of k between 0 and 1. What if k is negative or zero?

Rotations The matrix of a counterclockwise rotation in $\mathbb{R}^2$ through an angle θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Note that this matrix is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2 + b^2 = 1$. Conversely, any matrix of this form represents a rotation.

Example Examine how the linear transformation

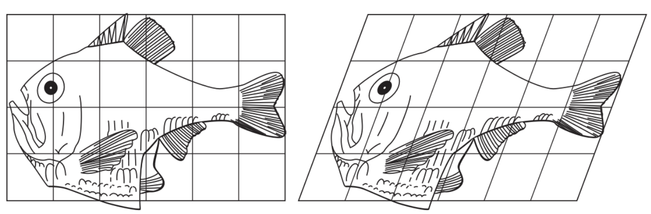
$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x}$$

affects our standard letter L. Here a and b are arbitrary constants.

Hint: Write an arbitrary vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in polar coordinates, as $\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$, where $r = \sqrt{a^2 + b^2}$, and examine how T affects it.

Horizontal and vertical shears

The matrix of a *horizontal shear* is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and the matrix of a *vertical shear* is of the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where k is an arbitrary constant. Here is an example of a horizontal shear that transforms the shape of one species of fish into another.



Argyropelecus olfersi. *Sternoptyx diaphana.*

Figure 16

[7] Thompson, d'Arcy W., *On Growth and Form*, Cambridge University Press, 1917. P. B. Medawar calls this "the finest work of literature in all the annals of science that have been recorded in the English tongue."

Example Sketch the image of the standard L under the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}$$

Example Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$